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Preface

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Abstract

The receptance matrix of a coupled system is derived in terms of the receptance properties of the individual component systems and coupling links. It is then shown that the resulting matrix equation may give quite inaccurate results near a resonance frequency of an individual component system where the receptance matrices for that component have large, nearly parallel columns. A formulation of the problem is given in terms of matrices with the large, parallel parts of the columns removed, thus avoiding the inaccuracy difficulty with very little additional computation. The formulation is valid for systems with damping, since all receptance terms are treated as complex numbers.

Receptance Coupling of Structural Components Near a Component Resonance Frequency

I. Introduction

The general method of characterizing the dynamic properties of linear systems by frequency-dependent ratios between inputs and outputs is now well known. The ratio factors are commonly put in matrix form and are variously called impedances, admittances or mobilities, receptances, four-pole parameters, transmission matrices, etc., depending on the particular type of ratio chosen.

If the dynamic properties of individual components of a complex system are known, then only matrix algebra is needed to determine the dynamic properties of the complex system. Although the matrix coupling operation is analytically straightforward, certain computational difficulties may arise in practice. In particular, each component system may have some resonance frequencies near which all ratios of response to excitation will become very large (or inverse ratios will become very small). In general, the resonance frequencies of the entire coupled system will not coincide with those of the individual components. Thus one may often find that "normal-size" dynamic characteristics (reflecting no resonance) must be computed by taking differences between very large terms that reflect an individual component system resonance. This report presents a formulation for greatly improving computational accuracy near a component-system resonance frequency. The receptance formulation of dynamic characteristics[1] is used.

II. The System

The total system considered in this study is made up of N component systems coupled together by n coupling links. The coupling links are considered to have flexibility and damping, but no mass. The Jth component system ($J = 1$ to N) has n_c^J points to which coupling links are attached, n_p^J points at which exciting forces are applied, and n_r^J points at which the displacement response is to be determined. These points will hereinafter be referred to as the *coupling points* (numbered from 1 to n_c^J), *excitation points* (1 to n_p^J), and the *response points* (1 to n_r^J) of component J. Each coupling link is attached to two coupling points (in two different components), so that n is related to the n_c^J values:

$$n = \frac{1}{2}\sum_{J=1}^{N} n_c^J \tag{1}$$

[1]E. Heer, "Coupled Systems Subjected to Determinate and Random Input," *Int. J. Solids Structures*, Vol. 3, pp. 155–166, 1967.

One assumes that certain matrices of receptance values for the component systems are known. In particular, assume that one knows the receptances necessary to write the column matrix of displacements at the response points in component J ($J = 1$ to N) as

$$\{X^J\} = [\mathcal{D}^J]\{P^J\} + [{}_*\bar{\mathcal{D}}^J]\{M^J\} \tag{2}$$

and the column matrix of coupling-point displacements as

$$\{D^J\} = [\bar{\mathcal{D}}^J]\{P^J\} + [\bar{\bar{\mathcal{D}}}^J]\{M^J\} \tag{3}$$

where $\{P^J\}$ is the column matrix of forces applied to the excitation points of component J and $\{M^J\}$ is the column matrix of forces applied to the coupling points of component J.

Equations (2) and (3) can also be expanded to describe the total system as

$$\{X\} = [\mathcal{D}]\{P\} + [{}_*\bar{\mathcal{D}}]\{M\} \tag{4}$$

and

$$\{D\} = [\bar{\mathcal{D}}]\{P\} + [\bar{\bar{\mathcal{D}}}]\{M\} \tag{5}$$

where the four column matrices can be partitioned as illustrated for $\{X\}$:

$$\{X\} = \begin{Bmatrix} X^1 \\ X^2 \\ \vdots \\ X^N \end{Bmatrix} \tag{6}$$

and the partitioning of the four rectangular receptance matrices is illustrated by $[\mathcal{D}]$:

$$[\mathcal{D}] = \begin{bmatrix} \mathcal{D}^1 & 0 & \cdots & 0 \\ 0 & \mathcal{D}^2 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{D}^N \end{bmatrix} \tag{7}$$

The elements of the rectangular receptance matrices in Eqs. (2) and (3) are, in general, complex and frequency-dependent. The elements of the column matrices can be either steady-state harmonic functions written as complex constants times $\varepsilon^{i\omega t}$ or as Fourier transforms of the time histories of the specified forces and displacements.

For the total system one seeks a rectangular ($n_x \times n_p$) receptance matrix $[H]$ that satisfies

$$\{X\} = [H]\{P\} \tag{8}$$

Note that

$$n_x = \sum_{J=1}^{N} n_x^J \tag{9}$$

and

$$n_p = \sum_{1}^{N} n_p^J \tag{10}$$

To obtain the matrix $[H]$ one must know the location and properties of the coupling links in addition to the component receptance matrices of Eqs. (2) and (3). To specify the location of coupling links, let $[C]$ be an ($n \times 2n$) rectangular matrix that can be partitioned as

$$[C] = [C^1 \,\vdots\, C^2 \,\vdots \cdots \vdots\, C^N] \tag{11}$$

where the $[C^J]$ submatrix is ($n \times n_c^J$). If coupling link l is attached to coupling point i in component I and coupling point j in component J, then either let

$$C^I_{li} = +1 \qquad \text{and} \qquad C^J_{lj} = -1 \tag{12}$$

or let

$$C^I_{li} = -1 \qquad \text{and} \qquad C^J_{lj} = +1 \tag{13}$$

Let all other elements of row l of $[C]$ be zero. Then row l of $[C]\{D\}$ is the relative displacement of the two ends of coupling link l. Let $\bar{M}_l$ be the force in coupling link l. (Since the link is massless, this force is the same at both ends of the link.) Since the system is linear, one can then write

$$[C]\{D\} = \lceil K \rfloor \{\bar{M}\} \tag{14}$$

where $\lceil K \rfloor$ is a diagonal matrix of complex frequency-dependent terms.

The column matrix of coupling-point forces $\{M\}$ is related to the column matrix of coupling-link forces $\{\bar{M}\}$ by the transpose of the $[C]$ matrix

$$\{M\} = [C]^T\{\bar{M}\} \tag{15}$$

The preceding equations are easily confirmed by analyzing Fig. 1 in which the extension of coupling link l is governed by

$$D^J_j - D^I_i = K_{ll}(\pm\bar{M}_l) \tag{16}$$

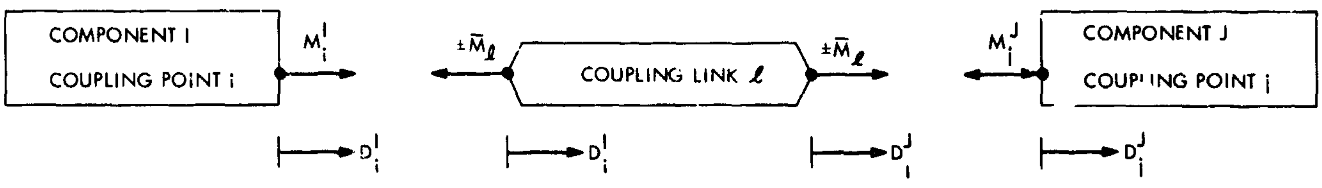

Fig. 1. Typical coupling link

Equating forces at each coupling point gives

$$-M_i^I = M_j^J = \pm\overline{M}_l \tag{17}$$

Using the plus sign for $\overline{M}_l$ in Eqs. (16) and (17) confirms Eqs. (14) and (15) with the row l elements of $[C]$ given by Eq. (13). Using the minus sign for $\overline{M}_l$ also confirms Eqs. (14) and (15), but this time with the $[C]$ elements given by Eq. (12). Thus Eqs. (12) and (13) are equivalent.

Combining Eqs. (5), (14), and (15) gives

$$[A]\{\overline{M}\} = [C][\overline{\mathcal{D}}]\{P\} \tag{18}$$

where

$$[A] = \lceil K \rfloor - [C][\overline{\overline{\mathcal{D}}}][C]^T \tag{19}$$

Letting $[A^{-1}]$ denote the inverse of $[A]$, one can then use Eq. (15) to substitute $\{\overline{M}\}$ from Eq. (18) into Eq. (4) to obtain

$$\{X\} = \left|[\mathcal{D}] + [*\overline{\mathcal{D}}][C]^T[A^{-1}][C][\overline{\mathcal{D}}]\right|\{P\} \tag{20}$$

Comparison of Eq. (20) with Eq. (8) shows that the desired $[H]$ matrix is given by

$$[H] = [\mathcal{D}] + [*\overline{\mathcal{D}}][C]^T[A^{-1}][C][\overline{\mathcal{D}}] \tag{21}$$

Because of the sparse nature of the receptance matrices illustrated by Eq. (7), one can partition $[H]$ in the obvious manner and obtain

$$[H^{IJ}] = [\mathcal{D}^{IJ}] + [*\overline{\mathcal{D}}^I][C^I]^T[A^{-1}][C^J][\overline{\mathcal{D}}^J] \tag{22}$$

where

$$[\mathcal{D}^{IJ}] \equiv [0] \qquad \text{for } I \neq J \tag{23}$$

and

$$[\mathcal{D}^{JJ}] \equiv [\mathcal{D}^J] \tag{24}$$

III. The Problem of Component Resonance

Note that in matrix $[A]$ the i, j component of the matrix $[C][\overline{\overline{\mathcal{D}}}][C]^T$ is the ratio between the relative displacement of the two attachment points of coupling link i and the force in coupling link j, $(\overline{M}_j)$, when all exciting forces are zero and all coupling forces except $\overline{M}_j$ are zero. For $i \neq i$, this relative displacement is an incompatibility of this single-force arrangement. For $i = j$, part of the relative displacement may be accommodated by the receptance of coupling link j, and the remainder is an incompatibility. Thus with $[A]$ defined by Eq. (19), the element A_{ij} is a measure of the incompatibility across coupling link i due to a force in coupling link j. The total column j of $[A]$ gives the incompatibilities across all the coupling links due to a force in coupling link j.

Suppose that a harmonic force with frequency near a resonance frequency of component R is applied to that component. The component response will be large, and for small damping it will be very nearly a pure mode shape. Thus the "shape" of the resonant component response will be only slightly affected by the location of the applied force. Hence, near a resonant frequency of component R ($R = 1$ to N), the rows and columns of $[A]$ that correspond to coupling links attached to component R will contain very large values. These columns with large values, though, will all be nearly parallel in the sense of vectors in n-dimensional space.

In particular, say that for some particular frequency one finds that all the component receptance terms for component R are $O(\lambda)$ where λ is large. Say that all the terms of $\lceil K \rfloor$ and of the other component receptance matrices are $O(1)$. Then by separating the contribution of the resonant mode of component R, one can write $[A]$ as a matrix with all $O(1)$ terms plus a matrix with all $O(\lambda)$ (and zero) terms. The columns of the $O(\lambda)$ matrix, though, will all be parallel.

Separating $[A]$ as suggested, one can easily show that $|A|$ is $O(\lambda)$. Similarly, the determinants of the cofactor matrices of $[A]$ are also $O(\lambda)$. Since the elements of $[A^{-1}]$ are ratios of cofactor determinants to $|A|$, it is obvious that $[A^{-1}]$ is $O(1)$.

Since $[A^{-1}]$ is $O(1)$ and the component receptance matrices of R are $O(\lambda)$, Eq. (22) gives $[H^{RJ}]$ and $[H^{IR}]$ as products of $O(1)$ matrices with an $O(\lambda)$ matrix. Further, Eq. (22) gives $[H^{RR}]$ as an $O(\lambda)$ matrix plus a product involving two $O(\lambda)$ matrices. However, since $[H]$ is the receptance matrix for the total coupled system, it should be $O(1)$ except near a resonant frequency of the total system. Thus, in general, Eq. (22) will result in calculating $O(1)$ terms as differences between $O(\lambda)$ terms for $[H^{RJ}]$ and $[H^{IR}]$. The computational difficulty of evaluating these $O(1)$ terms accurately is obvious.

IV. Alternative Form for Total System Receptances

A method of evaluating $[H]$ more accurately can best be seen by writing Eq. (22) in a somewhat different form. First one can write

$$[H^{IJ}] = [\mathcal{D}^{IJ}] + [*\bar{\psi}^{I}]\,[A^{-1}]\,[\bar{\psi}^{J}] \tag{25}$$

by defining two new matrices:

$$[\bar{\psi}^{J}] = [C^{J}]\,[\bar{\mathcal{D}}^{J}] \tag{26}$$

and

$$[*\bar{\psi}^{I}] = [*\bar{\mathcal{D}}^{I}]\,[C^{I}]^{T} \tag{27}$$

Column j of $[\bar{\psi}^{J}]$ contains n elements and gives in component J the incompatibilities at all the coupling points due to a unit force at excitation point j in component J (with all coupling forces equal to zero). Similarly, column i of $[*\bar{\psi}^{I}]$ gives the displacements at all the response points of component I due to a force in coupling link i.

As mentioned earlier, most of the above matrices are, in general, complex. One method of handling complex matrix arithmetic is to expand each matrix into a double-size real matrix. For example, let each of the rectangular matrices be replaced as follows:

$$[\bar{\bar{\mathcal{D}}}^{J}] = \left[\begin{array}{c|c} \bar{\bar{\mathcal{D}}}^{J}_{(R)} & -\bar{\bar{\mathcal{D}}}^{J}_{(I)} \\ \hline \bar{\bar{\mathcal{D}}}^{J}_{(I)} & \bar{\bar{\mathcal{D}}}^{J}_{(R)} \end{array}\right] \tag{28}$$

where the complex matrix is given by

$$[\bar{\bar{\mathcal{D}}}^{J}] = [\bar{\bar{\mathcal{D}}}^{J}_{(R)}] + \sqrt{-1}\,[\bar{\bar{\mathcal{D}}}^{J}_{(I)}] \tag{29}$$

and let each of the column matrices be replaced by

$$\{X^{I}\} = \left\{\begin{array}{c} X^{I}_{(R)} \\ \hline X^{I}_{(I)} \end{array}\right\} \tag{30}$$

where the complex form is

$$\{X^{I}\} = \{X^{I}_{(R)}\} + \sqrt{-1}\,\{X^{I}_{(I)}\} \tag{31}$$

Use of the double-size formulation for the matrices $[C^{J}]$ (with only zero imaginary parts) and the matrices $\lceil K \rfloor$, as well as all the component receptance matrices, results in no difference in form of any of the above matrix equations. The matrices $[A]$, $[A^{-1}]$, $[*\bar{\psi}^{I}]$, $[\bar{\psi}^{J}]$, and $[H^{IJ}]$ are now also double-size real matrices partitioned as in Eq. (28).

Using the double-size matrix formulation, one can expand Eq. (25) for the i,j term of $[H^{IJ}]$ as

$$H^{IJ}_{ij} = \mathcal{D}^{IJ}_{ij} + \sum_{k=1}^{2n}\sum_{l=1}^{2n} *\bar{\psi}^{I}_{ik}\,A^{-1}_{kl}\,\bar{\psi}^{J}_{lj} \tag{32}$$

for $i = 1$ to $2n^{I}_{r}$ and $j = 1$ to $2n^{J}_{p}$.

Alternatively, one can write H^{IJ}_{ij} as a ratio of determinants of real matrices:

$$H^{IJ}_{ij} = \frac{1}{|A|}\left|\begin{array}{c|ccc} \mathcal{D}^{IJ}_{ij} & *\bar{\psi}^{I}_{i1} & \cdots & *\bar{\psi}^{I}_{i,2n} \\ \hline \bar{\psi}^{J}_{1j} & & & \\ \vdots & & -[A] & \\ \vdots & & & \\ \bar{\psi}^{J}_{2n,j} & & & \end{array}\right| \tag{33}$$

The validity of Eq. (33) can be demonstrated by noting that expanding the larger determinant of Eq. (33) first by the upper row, then by the left-hand column, gives $\mathcal{D}^{IJ}_{ij}\,|-[A]|$ plus a double summation over k and l of $(-1)^{k+l+1} *\bar{\psi}^{I}_{ik}\,\bar{\psi}^{J}_{lj}$ times the determinant of $-[A]$ with column k and row l omitted. Since $[A]$ is $(2n \times 2n)$, one can see that $|-[A]| = |A|$, and $(-1)^{k+l+1}$ times the determinant of $-[A]$ with column k and row l omitted is the cofactor of element A_{lk} of $[A]$. Noting that $[A^{-1}]$ is equal to the transpose of the cofactor matrix of $[A]$ divided by $|A|$ completes the verification that Eq. (33) is equivalent to Eq. (32).

Note that for $1 \leq k \leq n$, column $k+1$ of the larger determinant in Eq. (33) gives either the real or the imaginary part of the displacement at response point i in component I, plus the real and imaginary parts of the incompatibility across each of the coupling links, all due to a force in coupling link k. Columns $n+2$ to $2n+1$ give a different arrangement of similar information. Column 1 gives either the real or the imaginary part of the displacement at response point i in component I, plus the real and imaginary parts of the incompatibility across each of the coupling links, all due to a force at excitation point j in component J.

Near a resonance frequency of component R ($R = 1$ to N), the large $O(\lambda)$ parts of columns 1 to n of the double-size $[A]$ matrix will be parallel. In addition, the $O(\lambda)$ parts of columns $n+1$ to $2n$ will be parallel. Similarly, for the larger matrix in Eq. (33), the $O(\lambda)$ parts of columns 2 to $n+1$ are parallel, the $O(\lambda)$ parts of columns $n+2$ to $2n+1$ are parallel, and the $O(\lambda)$ part of column 1 is parallel to the $O(\lambda)$ parts of either 2 to $n+1$ or $n+2$ to $2n+1$, depending on whether $l \leq n$ or $l > n$.

V. A Means of Avoiding the Component Resonance Problem

To evaluate the determinant of a matrix with nearly parallel columns, one can keep one column (say, column m) unchanged and subtract a multiple of column m from each of the other columns. To do this for the double-size real matrix $[A]$, it is advantageous to first choose m to correspond to one of the "longest" columns of $[A]$. In particular, let

$$E_k = \sum_{l=1}^{2n} A_{lk} A_{lk} \tag{34}$$

and let m be chosen so that

$$m \leq n \tag{35}$$

and

$$E_m \geq E_k \qquad \text{for } k = 1 \text{ to } n \tag{36}$$

Next, let

$$\alpha_m = 0 \tag{37}$$

and

$$\alpha_k = \frac{1}{E_m} \sum_{l=1}^{2n} A_{lk} A_{lm} \qquad \text{for } k \leq n \text{ and } k \neq m \tag{38}$$

Then for $k \leq n$ and $k \neq m$, α_k times column m is the component of column k that is parallel to column m. One can also show, from the partitioned form of the double-size $[A]$ matrix, that for $k > n$ and $k \neq m + n$, α_{k-n} times column $m + n$ is the component of column k that is parallel to column $m + n$.

Let a new matrix $[B]$ be defined by

$$B_{lk} = A_{lk} - \alpha_k A_{lm} \qquad \text{for } k \leq n \tag{39}$$

and

$$B_{lk} = A_{lk} - \alpha_{k-n} A_{l,m-n} \qquad \text{for } k > n \tag{40}$$

The matrix $[B]$ will then have only two columns (m and $m + n$) that are $O(\lambda)$. Since $[B]$ was formed by subtracting multiples of certain columns of $[A]$ from the other columns, one knows that

$$|B| = |A| \tag{41}$$

Similarly, define $[{*}\bar{\phi}^I]$ for $I = 1$ to N by

$${*}\bar{\phi}^I_{ik} = {*}\bar{\psi}^I_{ik} - \alpha_k \, {*}\bar{\psi}^I_{im} \qquad \text{for } k \leq n \tag{42}$$

and

$${*}\bar{\phi}^I_{ik} = {*}\bar{\psi}^I_{ik} - \alpha_{k-n} \, {*}\bar{\psi}^I_{i,m+n} \qquad \text{for } k > n \tag{43}$$

This removes the $O(\lambda)$ portions from all columns except m and $m + n$ of $[{*}\bar{\phi}^R]$.

To remove the $O(\lambda)$ portions from column 1 of the larger determinant in Eq. (33) for $J = R$, let

$$\beta^J_j = \frac{1}{E_m} \sum_{l=1}^{2n} \bar{\psi}^J_{lj} A_{lm} \tag{44}$$

Then define new matrix terms by

$$\bar{\phi}^J_{lj} = \bar{\psi}^J_{lj} - \beta^J_j A_{lm} \qquad \text{for } j \leq n^J_p \tag{45}$$

$$\bar{\phi}^J_{lj} = \bar{\psi}^J_{lj} - \beta^J_{j-n} A_{l,m-n} \qquad \text{for } j > n^J_p \tag{46}$$

$$\phi^{IJ}_{ij} = \mathcal{Q}^{IJ}_{ij} - \beta^J_j \, {*}\bar{\psi}^I_{im} \qquad \text{for } j \leq n^J_p \tag{47}$$

and

$$\phi^{IJ}_{ij} = \mathcal{Q}^{IJ}_{ij} - \beta^J_{j-n} \, {*}\bar{\psi}^I_{i,m+n} \qquad \text{for } j > n^J_p \tag{48}$$

Consider now the matrix

$$\left[\begin{array}{c|ccc} \phi^{IJ}_{ij} & {*}\bar{\phi}^I_{i1} & \cdots & {*}\bar{\phi}^I_{i,2n} \\ \hline \bar{\phi}^J_{1j} & & & \\ \vdots & & -[B] & \\ \bar{\phi}^J_{2n,j} & & & \end{array}\right] \tag{49}$$

In $\alpha_m = 0$, two columns of expression (49) are identical to the corresponding columns of the larger determinant of

Eq. (33); namely,

$$\text{column } m+1 = \begin{Bmatrix} *\bar{\psi}^{I}_{1m} \\ -A_{1m} \\ \cdot \\ \cdot \\ -A_{2n,m} \end{Bmatrix} \tag{50}$$

and

$$\text{column } m+n+1 = \begin{Bmatrix} *\bar{\psi}^{I}_{1,m+n} \\ -A_{1,m+n} \\ \cdot \\ \cdot \\ -A_{2n,m+n} \end{Bmatrix} \tag{51}$$

Further, for $k \leq n$ and $k \neq m$, column $k+1$ of expression (49) is formed by subtracting α_k times column $m+1$ from column $k+1$ of Eq. (33). Similarly, for $k > n$ and $k \neq m+n$, column $k+1$ of expression (49) is formed by subtracting α_k times column $m+n+1$ from column $k+1$ of Eq. (33). Finally, column 1 of expression (49) is formed by subtracting β^J_j times column $m+1$ from column 1 of Eq. (33) for $j \leq n^J_p$, and by subtracting β^J_j times column $m+n+1$ for $j > n^J_p$.

Thus the determinant of expression (49) is equal to the larger determinant of Eq. (33) and one can write

$$H^{IJ}_{ij} = \frac{1}{|B|} \left| \begin{array}{c|ccc} \phi^{IJ}_{ij} & *\bar{\phi}^{I}_{i1} & \cdots & *\bar{\phi}^{I}_{i,2n} \\ \hline \bar{\phi}^{J}_{1j} & & & \\ \vdots & & -[B] & \\ \bar{\phi}^{J}_{2n,j} & & & \end{array} \right| \tag{52}$$

By expanding by the upper row and the left-hand column (as in the verification of Eq. 33), one may write Eq. (52) as

$$H^{IJ}_{ij} = \phi^{IJ}_{ij} + \sum_{k=1}^{2n} \sum_{l=1}^{2n} *\bar{\phi}^{I}_{ik} B^{-1}_{kl} \bar{\phi}^{J}_{lj} \tag{53}$$

Thus one can write the matrix equation

$$[H^{IJ}] = [\phi^{IJ}] + [*\bar{\phi}^{I}] [B^{-1}] [\bar{\phi}^{J}] \tag{54}$$

If $[B]$ is partitioned into four $(n \times n)$ submatrices, one finds that the upper-left and lower-right submatrices are identical, and the upper-right submatrix is the same as the lower-left, except that the sign of each term is reversed. Thus the $(2n \times 2n)$ $[B]$ matrix can be considered as a double-size real version of an $(n \times n)$ complex matrix. As shown in Eq. (28), the real part of the $(n \times n)$ $[B]$ matrix will be the upper-left submatrix and the imaginary part will be the lower-left submatrix. The complex $[B]$ matrix can then be found directly from the complex $[A]$ matrix by use of Eq. (39).

Similarly, one can verify that the $[\phi^{IJ}]$, $[*\bar{\phi}^{I}]$, and $[\bar{\phi}^{J}]$ matrices can be considered as complex original-size matrices, which can be found directly from the complex $[\mathcal{Q}^{IJ}]$, $[*\bar{\psi}^{I}]$, and $[\bar{\psi}^{J}]$ matrices by use of Eqs. (47), (42), and (45).

Thus Eq. (54) can be considered as relating original-size complex matrices, rather than the double-size real matrices used in deriving it. The double-size formulation is convenient for analytical study, but not necessarily for numerical calculation of the complex matrix arithmetic. Since Eq. (54) is valid as an original-size complex matrix equation, one can use any suitable method for the numerical calculations.

Note that the α_k and β^J_j terms are defined in the double-size formulation. Equations (34), (38), and (44) can alternatively be written as summations from 1 to n, as follows:

$$E_k = \sum_{l=1}^{n} (A_{(R)lk} A_{(R)lk} + A_{(I)lk} A_{(I)lk}) \tag{55}$$

$$\alpha_k = \frac{1}{E_m} \sum_{l=1}^{n} (A_{(R)lk} A_{(R)lm} + A_{(I)lk} A_{(I)lm}) \qquad \text{for } k \neq m \tag{56}$$

and

$$\beta^J_j = \frac{1}{E_m} \sum_{l=1}^{n} (\bar{\psi}^{J}_{(R)lj} A_{(R)lm} + \bar{\psi}^{J}_{(I)lj} A_{(I)lm}) \tag{57}$$

The formulation given here verifies that Eq. (52) is equivalent to Eq. (33), but Eq. (52) can be evaluated more accurately, since $O(\lambda)$ parallel columns have been removed from the determinants. One can easily show that this improved accuracy also carries over to the matrix Eq. (54), which is equivalent to Eq. (22).

From the definitions of the new matrices, one finds that all elements of $[\phi^{IJ}]$ and $[\bar{\phi}^{J}]$ are $O(1)$ for I and $J = 1$ to N, in spite of some component resonance giving $C(\lambda)$ contributions to the original component receptance matrices. Similarly, the original-size $[*\bar{\phi}^{I}]$ for $I = 1$ to N and $[B]$ have all $O(1)$ terms, except in column m where they have $O(\lambda)$ terms. From the form of $[B]$ one knows that $|B|$ is $O(\lambda)$ and the cofactors of the elements of $[B]$ are $O(\lambda)$, except for elements in column m that have $O(1)$ cofactors.

Since $[B^{-1}]$ is the transpose of the cofactor matrix divided by $|B|$, one finds that $[B^{-1}]$ has all $O(1)$ terms except in row m, which is $O(1/\lambda)$. Thus all terms in Eq. (54) are $O(1)$ except $\bullet\bar{\phi}^{J}_{im}$ terms that are $O(\lambda)$ and B^{-1}_{mj} terms that are $O(1/\lambda)$. Expanding the matrix multiplication of Eq. (54) shows that a $\bullet\bar{\phi}^{J}_{im}$ term is always multiplied by a B^{-1}_{mj} term. Thus using Eq. (54) to evaluate $[H^{IJ}]$ involves additions of only $O(1)$ terms.

VI. Summary and Conclusions

The receptance matrix of a coupled system is derived in terms of the receptance properties of the individual component systems and coupling links. The result is Eq. (22), which is simple in form, but presents computational difficulties near a resonant frequency of one of the components. The difficulties arise from the fact that near a component resonant frequency the receptance matrices for that component have large, nearly parallel columns. Equations (39), (42), (45), and (47) define new matrices that have the large parallel parts of the columns removed. With the use of these new matrices, Eq. (54) is shown to be equivalent to Eq. (22), but without the component resonance computational difficulty. Systems with damping can be treated by this method, since all the receptances are considered as complex numbers.

Nomenclature

D^J_j	constrained displacement in system J at point j
M^J_j	constrained force in system J at point j
P^J_i	steady-state excitation in system J at point j
X^J_j	steady-state response in system J at point j
$[\bar{\bar{\mathcal{Q}}}^{J}]$	receptance matrix between coupling points in system J
$[\bar{\mathcal{Q}}^{J}]$	receptance matrix between a coupling point and an excitation point in system J
$[\bullet\bar{\mathcal{Q}}^{J}]$	receptance matrix between a coupling point and a response point in system J
$[\mathcal{Q}^{J}]$	receptance matrix between two field points in system J
[ ]	rectangular matrix
$[\ ]^T$	transpose of rectangular matrix
{ }	column matrix